Sharing Water in a Network*

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Abstract

We explore a game where agents choose a positive real number. This can be thought as a quantity of extracted water. Due to this extraction, the agents, which are embedded in a weighted and directed network, exert negative externalities on their immediate successors. This can be thought as water flow reduction for downstream neighbors; the higher the weight of the link, the higher the reduction. We characterize the Nash equilibrium and the social optimum profiles of this game as a function of the structure of the network. Then, we investigate policy measures that could improve welfare in this game either by imposing quotas, equal to equilibrium actions, or taxes, located around the spectral radius of the network.

Keywords: weighted directed network, strategic substitutes, negative externalities, optimal policies.

JEL: A13, D62, H23.

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1 Introduction

The present paper engages the classic question of restoring optimality in a strategic game played on a directed network, where players' actions are strategic substitutes to their neighbors' actions. This class of games, pioneered among others by Ballester et al. (2006), encompasses various well-known games including the voluntary contribution of local public goods (Bramoullé and Kranton, 2007; Bloch and Zenginobuz, 2007; Corbo et al., 2007). In such games, the relation between geographic (or social) structure and pure strategy Nash equilibrium is now well understood (see, e.g., Bramoullé et al., 2013; Rébillé and Richefort, 2014). This paper contributes to this literature by providing new knowledge of how geographic (or social) structure, social optimum and optimal policies are mutually related.

Here we consider a simultaneous-move game in which higher levels of action by neighbors lower an individual's payoff, in other words, we focus on directed network games of strategic substitutes and negative externalities. Typical application includes the extraction of water by a set of agents located on a river basin. Due to this extraction, the agents, which are embedded in a (weighted and directed) network of hydrological influences, exert local negative externalities by reducing water flow for their downstream neighbors (see, e.g., Ambec and Sprumont, 2002; Houba, 2008; Khmelnitskaya, 2010).[1] Another application is the production of a consumption good by a set of agents located on a common area (e.g., an irrigation area). Due to this production, the agents, which are embedded in a network of geographic proximities, exert local negative externalities by polluting their immediate neighbors (see, e.g., Janmaat, 2005; Ni and Wang, 2007). In both cases, the higher the weight of the link, the more the negative externality.

The main result is to establish the connection between efficiency-restoring policies and players' position in the network. To obtain this outcome, we first find sufficient conditions for the existence of a unique and interior Nash equilibrium.[2] We then find sufficient conditions for the existence of a unique and interior social optimum. Our proofs use standard optimization techniques

[1]In water resource economics, cooperative game theory has often been employed to analyze the problem of water resource allocation (Parrachino et al., 2006; Madani, 2010). In this literature, water is seen as a private good which can be traded and substituted with money (Ambec and Sprumont, 2002). Here, we investigate the problem of water allocation from a non-cooperative point of view because in many cases, participants may be reluctant to accept cooperative solutions requiring monetary transfers not based on market prices (Dinar et al., 1992).

[2]These results are in line with those obtained by Kaykobad (1985) on the existence of a positive steady state for a linear system of delay equations corresponding to the dynamics of n competing species.

as well as tools from the Perron-Frobenius theory of nonnegative matrices. Next, using modified versions of the Bonacich centrality measure, we define the *individual power* and the *social value* of a player in the network. With these definitions in hands, we show that

(i) players with a more "powerful" location in the network can exert a higher action;

(ii) players with a more "useful" location in the network should exert a higher action;

(iii) to restore efficiency, a lower quota and a higher tax rate should be imposed to players whose locations have a lower social value in the network.

The paper is built on the assumption of piece-wise linear best responses. In other words, a player's payoff function is separable into the linear benefit of action and the convex cost of action. The use of such a cost function is conventional in the economic theory of the commons. This reflects the idea that the resource has a scarcity value (Gordon, 1954; Smith, 1968).[3] Moreover, the use of a linear benefit function is standard in the economic theory of negative externalities. Individual preferences are therefore quasi-linear and explicit solutions on how to share and reduce the cost due to the external effects can be obtained (Baumol and Oates, 1988).

The main motivation of the model is to understand the extent to which geographic or social structures should be taken into account by policy makers in order to regulate wasteful behaviors towards common-pool resources. The role of geographic or social structure on the welfare loss caused by selfish use of a common good has been found by Johari and Tsitsiklis (2004), who study a network congestion game and show that the welfare of the equilibrium can be equal to 66 % of the optimum social welfare, and by İlkiliç (2010), who studies a water extraction game in bipartite networks and show that, under some structural conditions, water sources could be exploited above the efficient levels. Interestingly, the need for non-uniform incentives to restore optimality when people are organized into geographic or social "relationships" has been first established by Holmstrom (1982) when relationships induce positive externalities (the "public good" case), and by Segerson (1988) when relationships induce negative externalities (the "pollution" case). In a way, the present paper extends the "pollution" case by taking explicitely into account the *architecture* of the relationships.

[3] See Dinar et al. (1997) for a discussion on the use of convex cost functions in water resource economics.

2 Model

There are n players and the set of players is $N = \{1, \ldots, n\}$. Each player $l \in N$ chooses simultaneously an action $a_l \geq 0$. E.g., the players could be "agents" (municipalities, farmers, individuals, ...) extracting water and a_l could be the strategy of agent l's in the quantity extracted. We assume that player l's marginal benefit of action is $p_l > 0$. Let $\mathbf{a} = (a_1, \ldots, a_n)$ denote an action profile of all players.

Players are arranged in a network, which we represent as a weighted directed graph which consists of a set of nodes (the players), a set of arcs (the unidirectional externalities between players) and a mapping from the set of arcs to a set of positive weights (the intensities of the externalities). We will use kl to denote an arc directed from node k to node l. If kl is an arc, then we say that node l is a *downstream neighbor* (i.e., an immediate successor) of node k, or that node k is an *upstream neighbor* (i.e., an immediate predecessor) of node l. A directed path in the graph is a sequence of distinct nodes connected by arcs corresponding to the order of the nodes in the sequence. The length of a directed path is its number of arcs. The weight of a directed path is the product of the weights of its arcs. To continue our example, the network could reflect the hydrologic distribution of the agents on a river basin. Two agents are neighbors only when they are hydrologically dependent. In that case, the two agents are linked, the direction and the intensity of the link being determined by the flowing nature of water.

The basic representation of the graph is given by its weighted $n \times n$ adjacency matrix $\mathbf{\Omega} = [\omega_{kl}] \in \mathbb{R}_+^{n \times n}$ where $\omega_{kl} > 0$ if kl is an arc and $\omega_{kl} = 0$ otherwise (by convention $\omega_{kk} = 0$).

We suppose that players are "polluted" by their upstream neighbors' actions and hence that network externalities are negative. Let e_i denote player i's *environmental burden*, defined as the sum of player i's action level with the action levels of her upstream neighbors multiplied by the corresponding weights. Players incur the cost of their environmental burden according to a twice differentiable strictly convex cost function $q_l(e_l)$ defined on $\mathbb{R}_+$ with $q_l' > 0$ and $q_l'' > 0$ on $(0, \infty)$ for all l. Following our assumptions stated above, the resulting environmental burden is determined according to $e_l = a_l + \sum_{k:k \neq l} \omega_{kl} a_k$. E.g., $q_l(e_l)$ could be the value, in monetary units, to agent l of water extractions $\mathbf{a}$ of all agents.

The payoff function of player l, defined for all action profile $\mathbf{a} \geq \mathbf{0}$, is given by

$$U_l(\mathbf{a}) = p_l a_l - q_l \left(a_l + \sum_{k:k \neq l} \omega_{kl} a_k \right),$$

and we note $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ the simultaneous-move game with payoffs $U_l : \mathbb{R}_+^n \to \mathbb{R}$ and strategy space $a_l \geq 0$ for all l, where $\mathbf{p}$ is the vector of marginal benefits and $\mathbf{q}$ the vector of cost functions. Since $\partial U_l/\partial a_k \leq 0$ for all $k \neq l$, this is a game of negative externalities, and since $\partial^2 U_l/\partial a_k \partial a_l \leq 0$ for all $k \neq l$, this is a game of strategic substitutes. We focus on Nash equilibria in pure strategies.[4] For the rest of the paper, we require:

Assumption A0. $q_l'(0) < p_l < q_l'(\infty)$ for all $l \in N$.

We call this assumption the *boundary conditions.* If $q_l'(0) \geq p_l$, then player l would not exert any action and could be ignored, and if $q_l'(\infty) \leq p_l$, then player l's optimization problem has no solution.[5]

3 Equilibrium Profile

Given $\bar{z}_l = \sum_{k:k\neq l} \omega_{kl}\bar{a}_k$, each player l maximizes her payoff function with respect to own action constrained to be nonnegative. The first order conditions for a Nash equilibrium are

$$\begin{cases} \text{if } a_l > 0, & \text{then } p_l = q_l'\left(a_l + \bar{z}_l\right); \\ \text{if } a_l = 0, & \text{then } p_l \leq q_l'\left(a_l + \bar{z}_l\right). \end{cases}$$

It follows that an action profile $\hat{\mathbf{a}} \in \mathbb{R}_+^n$ is a Nash equilibrium of game $\mathcal{G}$ if and only if

$$\forall l \in N, \quad \hat{a}_l = \max\left\{0, a_l^* - \hat{z}_l\right\},$$

where $a_l^* = (q_l')^{-1}(p_l)$ is positive and finite (thanks to the boundary conditions), and $\hat{z}_l = \sum_{k:k\neq l} \omega_{kl}\hat{a}_k \geq 0$. From now on, we will refer to a_l^* as player l's *autarky equilibrium action* and for sake of simplicity, let $\mathbf{a}^* = (\mathbf{q}')^{-1}(\mathbf{p}) = ((q_l')^{-1}(p_l))_l$.

In network games, equilibrium analysis has traditionally been the primary research subject and great efforts have been made to increase our understanding of equilibrium behaviors and outcomes.[6] In network games of

[4] Note that the action space is the real line and the payoffs are strictly concave, so every mixed strategy is dominated, in terms of expected payoffs, by its average pure strategy.

[5] Under A0, we have $p_l - q_l'(0) > 0$ and $p_l - q_l'(\infty) < 0$ for all l. Moreover, p_l is a constant and q_l is strictly convex for all l. Thus, the existence of a Nash equilibrium profile is guaranteed by A0 and is a direct application of Theorem 1 in Rébillé and Richefort (2014).

[6] In particular, equilibrium existence, uniqueness and characterization in terms of players' network centrality has been well studied by several authors (Ballester et al., 2006; Bramoullé and Kranton, 2007; Bloch and Zenginobuz, 2007; Corbo et al., 2007; Ballester and Calvó-Armengol, 2010; Le Breton and Weber, 2011; Bramoullé et al., 2013; Rébillé and Richefort, 2014).

strategic substitutes, a sufficient uniqueness condition, that derives from contraction of the best response functions, states that $\rho(\mathbf{\Omega})$, the spectral radius of the network, should be sufficiently low.[7] This result has been recently improved: $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ admits a unique Nash equilibrium whenever $\mathbf{I} + \mathbf{\Omega}$ is a P-matrix.[8] However, these conditions do not allow to distinguish between corner, partially-corner and interior equilibria. We contribute to this literature by focusing on interior Nash equilibria. We require:

Assumption A1. $\left(\mathbf{I} - \mathbf{\Omega}^{\mathsf{T}}\right) \mathbf{a}^* >> \mathbf{0}$.

This assumption guarantees that for each player, the difference between her own autarky equilibrium action and the sum of her upstream neighbors' autarky equilibrium actions is always positive.[9] We obtain the following result.

Proposition 1. *Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game. Under A0 and A1, the equilibrium profile exists, is unique and is interior.*

This result establishes the sharpest sufficient condition for the existence (guaranteed by A0) of a unique (guaranteed by A1 since A1 $\Rightarrow \rho(\mathbf{\Omega}) < 1 \Rightarrow$ $\mathbf{I}+\mathbf{\Omega}$ is a P-matrix) and interior Nash equilibrium in directed network games of strategic substitutes and negative externalities.

We now look for a closed-form solution to the equilibrium problem. For this purpose, we introduce a modified version of the Bonacich centrality measure. For a weighted adjacency matrix $\mathbf{\Omega} \in \mathbb{R}_+^{n \times n}$ and for a vector of weights $\mathbf{v} \in \mathbb{R}_+^n$ assigned to the nodes, the (weighted) Bonacich centrality measure is given by

$$\mathbf{c}\left(\alpha, \beta, \mathbf{\Omega}, \mathbf{v}\right) = \alpha \left(\mathbf{I} - \beta \mathbf{\Omega}\right)^{-1} \mathbf{\Omega} \mathbf{v},$$

where $\alpha, \beta \in \mathbb{R}$ are two scalars (Bonacich, 1987). If $|\beta|$ is sufficiently low, we have the following power expansion:

$$\mathbf{c}\left(\alpha, \beta, \mathbf{\Omega}, \mathbf{v}\right) = \alpha \sum_{k=0}^{\infty} \beta^k \mathbf{\Omega}^{k+1} \mathbf{v},$$

[7] In undirected networks, the spectral radius is a standard measure of the density of the network. See, e.g., Cvetkovic and Rowlinson (1990).

[8] See Rébillé and Richefort (2014) for the proof and for the geometric/economic interpretation of this result.

[9] Assumption A1 can be related to Assumption (2.4) in Kaykobad (1985)'s paper on the existence of a positive solution for a linear nonhomogeneous system of equations with positive coefficients.

so the Bonacich centrality measure counts the total weight of all directed paths ending at each node in the network. Moreover, when β is negative, even length directed path are weighted negatively and odd length directed path are weighted positively, hence $\mathbf{c}(1, \beta, \mathbf{\Omega}, \mathbf{1})$ measures the (bargaining) *power* of a node (Bonacich, 1987, p. 1176). We adapt this definition of power to our framework.

Definition 1. Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game. If $\mathbf{I} + \mathbf{\Omega}^{\mathsf{T}}$ is invertible, the vector

$$\mathbf{b}_{\text{alt}}^{-}(\mathbf{\Omega}, \mathbf{a}^*) = \mathbf{a}^* - \mathbf{c}\left(1, -1, \mathbf{\Omega}^{\mathsf{T}}, \mathbf{a}^*\right) = \left(\mathbf{I} + \mathbf{\Omega}^{\mathsf{T}}\right)^{-1} \mathbf{a}^*$$

is called the *individual power* measure.

The individual power of a player in the network is a sum of her autarky equilibrium action with the total weight of all directed paths that end at her, where odd length directed paths are weighted negatively and even length directed paths are weighted positively, and where a directed path that starts at player k is weighted by a_k^*, the autarky equilibrium action of the corresponding player. Hence, having many upstream neighbors reduces individual power, but if one player's upstream neighbors themselves have many upstream neighbors, individual power is increased, and so on.

Proposition 2. *Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game. Under A0 and A1, the equilibrium profile is given by*

$$\hat{\mathbf{a}} = \mathbf{b}_{\text{alt}}^{-}(\mathbf{\Omega}, \mathbf{a}^*).$$

When A0 and A1 are met, the unique Nash equilibrium is interior. In this case, players that have a higher individual power in the network have more powerful locations and consequently, they can exert a higher action.[10]

Example 1 (Quadratic costs and homogeneous benefits). Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game. Assume $U_l(a_l, e_l) = pa_l - \frac{\gamma_l}{2} e_l^2$ for all l where $p, \gamma_l > 0$ (then A0 is met and $a_l^* = \frac{p}{\gamma_l} > 0$). Thus, A1 becomes

$$\left(\mathbf{I} - \mathbf{\Omega}^{\mathsf{T}}\right) \frac{\mathbf{1}}{\boldsymbol{\gamma}} >> \mathbf{0}$$

where $(\frac{\mathbf{1}}{\boldsymbol{\gamma}})_l = \frac{1}{\gamma_l}$ for all l. When this condition is satisfied, we obtain

$$\hat{\mathbf{a}} = p\ \mathbf{b}_{\text{alt}}^{-}\left(\mathbf{\Omega}, \frac{\mathbf{1}}{\boldsymbol{\gamma}}\right).$$

[10]In other words, the equilibrium action of a player decreases with odd length directed paths and increases with even length directed paths that end at her in the network. If there is a directed path of odd length (resp. even length) from player k to player l, player l's action is a strategic substitute (resp. strategic complement) of player k's action. Otherwise, their actions are independent.

4 Efficient Profile

For the analysis of the efficient action profile, we adopt a standard utilitarian approach.[11] Let SW be the social welfare function defined for all $\mathbf{a} \geq \mathbf{0}$ by

$$SW\left(\mathbf{a}\right) = \sum_{l} \left[p_l a_l - q_l \left(a_l + \sum_{k:k\neq l} \omega_{kl} a_k \right) \right].$$

Given $\boldsymbol{\Omega}$, $\mathbf{p}$ and $\mathbf{q}$, an action profile is said to be *efficient*, or *socially optimal*, if it maximizes the social welfare function. The first order conditions for an efficient action profile are

$$\begin{cases} \text{if } a_l > 0, \quad \text{then } p_l - q_l' \left(a_l + \sum\limits_{k:k\neq l} \omega_{kl} a_k \right) = \sum\limits_{j:j\neq l} \omega_{lj} q_j' \left(a_j + \sum\limits_{i:i\neq j} \omega_{ij} a_i \right); \\ \text{if } a_l = 0, \quad \text{then } p_l - q_l' \left(a_l + \sum\limits_{k:k\neq l} \omega_{kl} a_k \right) \leq \sum\limits_{j:j\neq l} \omega_{lj} q_j' \left(a_j + \sum\limits_{i:i\neq j} \omega_{ij} a_i \right). \end{cases}$$

Hence, an efficient action profile $\tilde{\mathbf{a}} \in \mathbb{R}_+^n$ satisfies, for all l,

$$\tilde{a}_l > 0 \iff p_l - q_l'\left(\tilde{z}_l\right) > \sum_{j:j\neq l} \omega_{lj} q_j'\left(\tilde{a}_j + \tilde{z}_j\right).$$

where $\tilde{z}_l = \sum_{k:k\neq l} \omega_{kl} \tilde{a}_k$ and $\tilde{z}_j = \sum_{i:i\neq j,l} \omega_{ij} \tilde{a}_i$.

In such games, welfare analysis has not been investigated so much, although this subject is crucial to understanding the upper bounds on the network's performance.[12] In the present paper, we investigate the existence, the uniqueness, the interiority and the characterization of the efficient action profile for general network structures (acyclic or not, directed or not, weighted or not). We require:

Assumption A2. $\left(\mathbf{I} - \boldsymbol{\Omega}\right)\mathbf{p} - \mathbf{q}'\left(\boldsymbol{\Omega}^{\mathsf{T}}\mathbf{a}^*\right) >> \mathbf{0}$, where $p_l = q_l'(a_l^*)$ for all $l \in N$.

[11] This is for ease of exposition. This also reflects the interest of the social planner for the various players without (geographic, social, ...) discrimination. However, one can easily check that all our results extend to weighted social welfare functions.

[12] A natural question already considered in the literature has been to identify the Nash equilibrium maximizing social welfare. In particular, positive and negative effects of removing a player (Ballester et al., 2006), adding a new link (Bramoullé and Kranton, 2007; Bramoullé et al., 2013) or changing the intensity of a link (Bloch and Zenginobuz, 2007) have been analyzed. Another attempt at analyzing social welfare can be found in Rébillé and Richefort (2012) for acyclic networks.

This assumption guarantees that for each player, the difference between her own marginal benefit and the sum of her downstream neighbors' marginal benefits is always greater than the marginal cost of the maximal negative impact (given by the autarky equilibrium action) caused by her upstream neighbors. We obtain the following result.

Proposition 3. *Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game.*

(i) Under A0 and A2, the efficient profile exists, is unique and is interior.

(ii) Moreover, A2 $\Rightarrow$ A1 $\Rightarrow \mathbf{I} + \mathbf{\Omega}$ is invertible $\Rightarrow \tilde{\mathbf{a}} \in \mathbb{R}^n_+$ is unique.

This result establishes the sharpest sufficient condition for the existence (guaranteed by A0) of a unique (interior if A2 is met; corner or partially-corner otherwise) social optimum in directed network games of strategic substitutes and negative externalities. We now look for a closed-form solution to the efficiency problem. For this purpose, we introduce another modified version of the weighted Bonacich centrality measure.

Definition 2. Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game. If $\mathbf{I} + \mathbf{\Omega}$ is invertible, the vector

$$\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega}, \mathbf{p}) = \mathbf{p} - \mathbf{c}(1, -1, \mathbf{\Omega}, \mathbf{p}) = (\mathbf{I} + \mathbf{\Omega})^{-1} \mathbf{p}$$

is called the *social value* measure.

The social value of a player in the network is a sum of her marginal benefit of action with the total weight of all directed paths that start at her, where odd length directed paths are weighted negatively and even length directed paths are weighted positively, and where a directed path that ends at player j is weighted by p_j, the marginal benefit of action of the corresponding player. Hence, having many downstream neighbors reduces social value, but if one player's downstream neighbors themselves have many downstream neighbors, social value is increased, and so on.

From now on, we will refer to $\tilde{a}^*_l = (q'_l)^{-1}((\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega}, \mathbf{p}))_l)$ as player l's *autarky efficient action* and for sake of simplicity, let $\tilde{\mathbf{a}}^* = (\mathbf{q}')^{-1}(\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega}, \mathbf{p}))$.

Proposition 4. *Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game. Under A0 and A2, the efficient profile is given by*

$$\tilde{\mathbf{a}} = \mathbf{b}^{-}_{\text{alt}}(\mathbf{\Omega}, \tilde{\mathbf{a}}^*).$$

When A0 and A2 are met, the unique social optimum is interior and is equal to a combination of the individual power and the social value measures. This combination may reflect the *social power* of the players. Hence, players that have a higher social power in the network have more socially powerful locations and consequently, they should exert a higher action.[13] Furthermore, the higher the social value of a player, the higher her autarky efficient action and therefore, the higher her efficient action. Thus, players that have a higher social value have more useful locations and consequently, they should exert a higher action.

Example 2 (Quadratic costs and homogeneous benefits continued). Since $U_l(a_l, e_l) = pa_l - \frac{\gamma_l}{2}e_l^2$ for all l, A2 becomes

$$\left(\mathbf{I} - \mathbf{\Omega} - \mathbf{\Omega}^{\mathsf{T}}\right) \frac{\mathbf{1}}{\gamma} >> \mathbf{0}.$$

When this condition is met, we obtain

$$\tilde{\mathbf{a}} = p\ \mathbf{b}^{-}_{\text{alt}}\left(\mathbf{\Omega}, \mathbf{b}^{+}_{\text{alt}}\left(\mathbf{\Omega}, \frac{\mathbf{1}}{\gamma}\right)\right).$$

5 Optimality-Restoring Policies

We are now interested in deriving policies that restore optimality in games with local substitutabilities and negative externalities. This purpose is important because in such games, the equilibrium action profile is always inefficient and therefore, there are losses in social welfare.

Firstly, we focus on a situation where players' actions are constrained by a quota. The payoff of a player l is then given by

$$U_l(\mathbf{a}) = p_l a_l - q_l \left(a_l + \sum_{k:k\neq l} \omega_{kl} a_k\right),$$

with $a_l \leq \kappa_l$ for all l, where κ_l is player l's action quota.

Proposition 5. *Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game. Under A0 and A2, the optimal vector of quotas is given by*

$$\tilde{\boldsymbol{\kappa}} = \tilde{\mathbf{a}} = \mathbf{b}^{+}_{\text{alt}}\left(\mathbf{\Omega}, \tilde{\mathbf{a}}^{*}\right).$$

As it turns out, the optimal vector of quotas is actually the efficient action profile. The interpretation is straightforward. Players that have a

[13] In other words, the efficient action of a player increases with even length directed paths and decreases with odd length directed paths that end and start at her in the network. If there is a directed path of odd length (resp. even length) from player k to player l, their actions are negatively (resp. positively) correlated. Otherwise, their actions are independent.

higher social power should exert a higher action and therefore, they should get a higher quota. In other words, a lower quota should be imposed to players whose locations have a lower social value.

Secondly, we provide a tax mechanism which achieves the efficient action profile at game $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$. The mechanism penalizes players for their deviations from the efficient actions and hence, players prefer to exert the efficient action level. The payoff of a player l is then given by

$$U_l(\mathbf{a}) = p_l (1 - \tau_l) a_l - q_l \left(a_l + \sum_{k:k \neq l} \omega_{kl} a_k \right),$$

where $\tau_l \in [0, 1)$ is player l's tax rate on benefits.

Proposition 6. *Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game. Under A0 and A2, the optimal tax rates satisfy the following assertions.*

(i) $\forall l \in N$,

$$\tilde{\tau}_l = 1 - \frac{\left(\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega}, \mathbf{p})\right)_l}{p_l}, \quad \textit{with } \tilde{\tau}_l \in [0, 1).$$

(ii) $\forall l \in N$,

$$\tilde{\tau}_l > 0 \iff l \textit{ has a downstream neighbor.}$$

The higher the social value of a player in the network, the lower her optimal tax rate on benefits. Hence, the optimal tax rates depends on players' position in the network: they reflect both the marginal damages and the marginal benefits players produce on other players at the efficient profile.[14] Moreover, the optimal tax rates are always between zero (included) and one (excluded). In particular, $\tilde{\tau}_l = 0$ if and only if player l has no downstream neighbors in the network.[15] The following result provides a deeper understanding of how network structure can shape the optimal tax rates.

Proposition 7. *Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game. Under A0 and A2, the optimal tax rates satisfy the following inequality:*

$$\tilde{\tau}_{\min} \leq \frac{\rho(\mathbf{\Omega})}{1 + \rho(\mathbf{\Omega})} \leq \tilde{\tau}_{\max},$$

where $\tilde{\tau}_{\min} = \min_l\{\tilde{\tau}_l\}$ *and* $\tilde{\tau}_{\max} = \max_l\{\tilde{\tau}_l\}$.

This result states that the optimal tax rates are *centered* on an increasing

[14] More precisely, the optimal tax rate imposed to a player is positively related with the total weight of odd length directed paths and negatively related with the total weight of even length directed paths that start at her, where a directed path that end at player j is weighted by p_j.

[15] The proof relies on the fact that $\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega}, \mathbf{p})$ is always positive whenever A2 is satisfied.

function of $\rho(\mathbf{\Omega})$. This implies, for instance, that if the network is acyclic[16], there exists l such that $\tilde{\tau}_l = 0$. Following Proposition 6, we know that these tax rates down to zero concern players who have no downstream neighbors in the network.[17] If $\rho(\mathbf{\Omega}) > 0$, there exists l such that $\tilde{\tau}_l > 0$. Hence, whenever the optimal tax rate is uniform, it must be related to the spectral radius of the network.

Proposition 8. *Let $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game. Under A0 and A2, the following results hold.*

(i) If the optimal tax rate is uniform, then $\mathbf{p}$ is an eigenvector of $\mathbf{\Omega}$.

(ii) Conversely, if $\mathbf{p}$ is an eigenvector of $\mathbf{\Omega}$ and $t_{\mathbf{p}}$ its associated eigenvalue, then $t_{\mathbf{p}} \geq 0$ and $\tau = \frac{t_{\mathbf{p}}}{1+t_{\mathbf{p}}} \in [0, 1)$ is the uniform optimal tax rate.

(iii) Moreover, the optimal uniform tax rate is given by

$$\tilde{\tau} = \frac{\rho(\mathbf{\Omega})}{1 + \rho(\mathbf{\Omega})}.$$

This result shows that the optimal tax rate is uniform if and only if the vector of marginal benefits is an eigenvector of the network. In general, however, this condition is hardly met. Then, the optimal policy almost surely implies a discrimination between players according to their position in the network. But when $\mathbf{p}$ is an eigenvector of the network[18], the optimal tax rate is uniform and is equal to the increasing function of $\rho(\mathbf{\Omega})$ used in Proposition 7 to center the optimal tax rates.[19]

Example 3 (Quadratic costs and homogeneous benefits continued). Since $U_l(a_l, e_l) = pa_l - \frac{\gamma_l}{2}e_l^2$ for all l, we obtain

$$\tilde{\boldsymbol{\tau}} = \mathbf{1} - \mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega}, \mathbf{1}),$$

[16] In that case, $\rho(\mathbf{\Omega}) = 0$ (see Rébillé and Richefort, 2012).

[17] At least one such player always exists when the network is acyclic.

[18] Such a situation arises, for instance, when marginal benefits are uniform and players have the same weighted out-degree.

[19] Interestingly, observe that

$$\frac{\rho(\mathbf{\Omega})}{1 + \rho(\mathbf{\Omega})} = \rho\left(\sum_{k=0}^{\infty} (-1)^k \mathbf{\Omega}^{k+1}\right) = \rho(\mathbf{M}).$$

Thus, the optimal uniform tax rate is equal to the spectral radius of a matrix $\mathbf{M} = \mathbf{c}(1, -1, \mathbf{\Omega}, .)$ whose entry m_{kl} counts the total weight of all directed paths in the network starting at player k and ending at player l, where odd length directed paths are weighted positively and even length directed paths are weighted negatively.

whenever A2 is met. Suppose that players are arranged either along a “river” or around a “lake” (see Figure 1 below). In both cases, let the same weight $\delta \in (0,1)$ be attached to each arc and assume that δ is sufficiently low to satisfy A2.

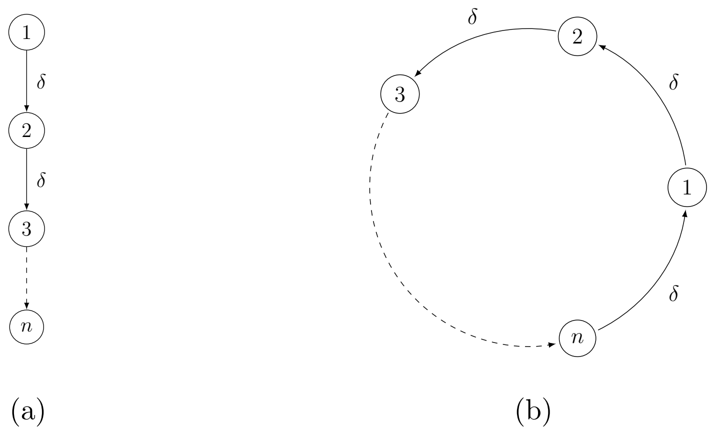

Figure 1: Geographic/Hydrologic structures with n players

Let us compute the optimal tax rates for these two classic networks.

(a) *The river.* This case illustrates Propositions 6 and 7. Player n is located at the tail end of the river, she has no downstream neighbors and therefore, $\tilde{\tau}_n = \tilde{\tau}_{\min} = 0$. All other optimal tax rates are positive, and since players are heterogeneous in their position along the river network, these tax rates are non-uniform. We obtain:

$$\forall l \in N, \;\; \tilde{\tau}_l = \frac{\delta + (-\delta)^{n-l+1}}{1+\delta}.$$

Moreover, $\tilde{\tau}_{\max} = \tilde{\tau}_{n-1} = \delta \; (< 1)$. Thus, $\tilde{\tau}_l \in [0, \delta]$ for all l.

(b) *The lake.* This case illustrates Proposition 8. We have $\mathbf{\Omega p} = \delta \mathbf{p}$, so $\mathbf{p}$ is an eigenvector of $\mathbf{\Omega}$ and δ its associated eigenvalue. Since $\mathbf{p} >> \mathbf{0}$, it follows that $\rho(\mathbf{\Omega}) = \delta$ and therefore, the optimal tax rate is uniform (though players have heterogeneous cost functions). We obtain:

$$\forall l \in N, \;\; \tilde{\tau}_l = \frac{\delta}{1+\delta}.$$

When $n \to \infty$, the river can be considered as a lake: the term $(-\delta)^{n-l+1}$ tends to zero (since $\delta < 1$) and the optimal tax rates of the river give the uniform optimal tax rate of the lake.

6 Extension

We show that our main insights can be used for establishing a necessary and sufficient condition under which each player exerts a strictly higher level of action than her efficient level. At the end of the appendix, we also discuss how our policy results would be affected by relaxing the focus on interior solutions.

Following Hardin (1960)'s terminology, the situation in which each player exerts a strictly higher level of action than her efficient level is called a Tragedy of the Commons, and we investigate the relation between such a situation and network structure. According to Proposition 6, we have:

$$\forall l \in N, \;\; \tilde{\tau}_l = 1 - \frac{\left(\mathbf{b}^{+}_{\text{alt}}\left(\boldsymbol{\Omega}, \mathbf{p}\right)\right)_l}{p_l}, \;\; \text{with } \tilde{\tau}_l \in [0,1).$$

Hence, $\mathbf{b}^{+}_{\text{alt}}(\boldsymbol{\Omega}, \mathbf{p}) = (\mathbf{1} - \boldsymbol{\tau}) \times \mathbf{p}$ and $\tilde{\mathbf{a}} = \mathbf{b}^{-}_{\text{alt}}(\boldsymbol{\Omega}, (\mathbf{q}')^{-1}((\mathbf{1} - \boldsymbol{\tau}) \times \mathbf{p}))$. We may now consider the difference vector $\mathbf{d} = \hat{\mathbf{a}} - \tilde{\mathbf{a}}$. We have

$$\mathbf{d} = \mathbf{b}^{-}_{\text{alt}}\left(\boldsymbol{\Omega}, \mathbf{a}^*\right) - \mathbf{b}^{-}_{\text{alt}}\left(\boldsymbol{\Omega}, \tilde{\mathbf{a}}^*\right) = \mathbf{b}^{-}_{\text{alt}}\left(\boldsymbol{\Omega}, \mathbf{d}^*\right)$$

where $\mathbf{d}^* = \mathbf{a}^* - \tilde{\mathbf{a}}^*$. By Proposition 6, we know that $\mathbf{d}^* >> \mathbf{0}$ whenever $\tilde{\boldsymbol{\tau}} >> \mathbf{0}$ (since $(\mathbf{q}')^{-1}$ is increasing by A0 and $\boldsymbol{\tau} \geq \mathbf{0}$). That is, $\mathbf{a}^* >> \tilde{\mathbf{a}}^*$ whenever each player has at least one downstream neighbor in the network, in other words, if there are no "sink" players.

Property 1. *Let $\mathcal{G}(\boldsymbol{\Omega}, \mathbf{p}, \mathbf{q})$ be a network game. Then a Tragedy of the Commons occurs, i.e.,*

$$\hat{\mathbf{a}} >> \tilde{\mathbf{a}}$$

if and only if the following linear system of inequalities

$$\begin{cases} \left(\mathbf{I} + \boldsymbol{\Omega}^{\mathsf{T}}\right)\mathbf{d} & = & \mathbf{d}^* \\ \mathbf{d} & >> & \mathbf{0} \end{cases}$$

admits a solution. In particular, if $\left(\mathbf{I} - \boldsymbol{\Omega}^{\mathsf{T}}\right)\mathbf{d}^* >> \mathbf{0}$ holds.

Note that whenever a Tragedy of the Commons occurs, then necessarily the network has no sink players. Indeed, if the system described in Property 1 admits a solution $\mathbf{d} >> \mathbf{0}$, then $\mathbf{d}^* >> \mathbf{0}$, so $\boldsymbol{\tau} >> \mathbf{0}$, thus by Proposition 6 the network possesses no sink players.[20]

[20] The sufficient condition we propose here is an application of Assumption A1 in Proposition 1. This condition is also related to Kaykobad (1985)'s existence result for a positive solution of positive linear systems.

7 Conclusion

This paper brings a social welfare analysis to directed network games of strategic substitutes and negative externalities, and is the first to establish the relation between geographic (or social) structure, social optimum and optimal policies.

Precisely, we highlight the implications of the upstream-downstream relationships on outcomes of the game: we formulate the equilibrium, the social optimum and the optimal policies profiles in terms of the directed paths that end and start at each player in the network, where directed paths of even and odd length have opposing signs in the closed-form expressions. These findings are consistent with previous results in the literature on the commons, in particular on the problem of efficient water resource allocation, whether from cooperative point of view (see, e.g., Ambec and Sprumont, 2002) or non-cooperative point of view (see, e.g., İlkiliç, 2010).

The cost function we consider allows us to investigate local negative externalities in games of strategic substitutes. A player's cost function depends on own action and the action of all her upstream neighbors. Our cost function is convex and allows for asymmetric effects between upstream neighbors' action and own action. Thus, our model can be interpreted in terms of overuse of a common good (e.g., water) or in terms of pollution externalities (see, e.g., Janmaat, 2005; Ni and Wang, 2007). In addition, benefits are linear in (and only depends on) own action. This specification of preferences, although restrictive, helps us focus on the effects of network structure at equilibrium, at efficiency and on optimal policies.

This work creates space for further research. The optimal tax plan designed in this paper raises issues as to the how social welfare, as well as property rights, should be defined in a social system. A further issue for investigation is how to redistribute the revenue generated by optimal taxes. Other extensions concern some refinements of the model that could be undertaken. For instance, actions could be constrained, at the individual level and/or at the system level. This should lead us to incorporate dynamics and stock issues in the model. Finally, it would also be pertinent to test the robustness of our policy results to more general specifications of preferences. The case of additive separable utility functions could be a reasonable first step towards this goal.

Appendix

A. Proofs of the results

Proof of Proposition 1. Existence. See footnote 5.

Uniqueness. A Nash equilibrium $\hat{\mathbf{a}} \in \mathbb{R}^n_+$ of game $\mathcal{G}(\boldsymbol{\Omega}, \mathbf{p}, \mathbf{q})$ is solution of the Linear Complementarity Problem

$$\begin{cases} \left(\mathbf{I} + \boldsymbol{\Omega}^\mathsf{T}\right)\hat{\mathbf{a}} & \geq & \mathbf{a}^* \\ \hat{\mathbf{a}} & \geq & \mathbf{0} \\ \hat{\mathbf{a}}^\mathsf{T}\left[\left(\mathbf{I} + \boldsymbol{\Omega}^\mathsf{T}\right)\hat{\mathbf{a}} - \mathbf{a}^*\right] & = & \mathbf{0} \end{cases} \tag{1}$$

Theorem 2 in Rébillé and Richefort (2014) entails that (1) admits a unique solution whenever $\mathbf{I} + \boldsymbol{\Omega}$ is a P-matrix (see Fiedler and Pták, 1962). Let us show that A1 implies that $\mathbf{I} + \boldsymbol{\Omega}$ is a P-matrix. We note $\lambda^{\mathbf{R}}_{\min}(\boldsymbol{\Omega})$ the lowest real eigenvalue of $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}_{I\times J}$ the submatrix of $\boldsymbol{\Omega}$ with rows in $I \subseteq N$ and columns in $J \subseteq N$.

Under A1, $\boldsymbol{\Omega}$ is a productive matrix (see Gale, 1960), so for all $I \subset N$, $\boldsymbol{\Omega}_{I\times I}$ is also a productive matrix. Then, $\rho(\boldsymbol{\Omega}_{I\times I}) < 1$ for all $I \subseteq N$ (see Berman and Plemmons, 1994). Hence, $\lambda^{\mathbf{R}}_{\min}(\boldsymbol{\Omega}_{I\times I}) > -1$ and therefore $\lambda^{\mathbf{R}}_{\min}((\mathbf{I} + \boldsymbol{\Omega})_{I\times I}) > 0$ for all $I \subseteq N$. So $\mathbf{I} + \boldsymbol{\Omega}$ is a P-matrix.

Interiority. Consider a unique Nash equilibrium $\hat{\mathbf{a}}$. By construction, for every l it holds

$$\hat{a}_l = \max\left\{0, a^*_l - \hat{\mathbf{a}}^\mathsf{T}\boldsymbol{\Omega}_{.l}\right\} \geq 0$$

and since $\boldsymbol{\Omega} \geq \mathbf{0}$,

$$\hat{a}_l = \max\left\{0, a^*_l - \hat{\mathbf{a}}^\mathsf{T}\boldsymbol{\Omega}_{.l}\right\} \leq a^*_l.$$

Hence, $0 \leq \hat{\mathbf{a}} \leq \mathbf{a}^*$ and we have for every l,

$$\hat{a}_l = \max\left\{0, a^*_l - \hat{\mathbf{a}}^\mathsf{T}\boldsymbol{\Omega}_{.l}\right\} \geq \max\left\{0, a^*_l - {\mathbf{a}^*}^\mathsf{T}\boldsymbol{\Omega}_{.l}\right\} > 0, \text{ by A1.}$$

Thus, $\hat{\mathbf{a}}$ is interior. □

Proof of Proposition 2. Under A0 and A1, (1) becomes

$$\begin{cases} \left(\mathbf{I} + \boldsymbol{\Omega}^\mathsf{T}\right)\hat{\mathbf{a}} & = & \mathbf{a}^* \\ \hat{\mathbf{a}} & >> & \mathbf{0} \end{cases}$$

Since A1 $\Rightarrow \rho(\boldsymbol{\Omega}) < 1$ (Gale, 1960; Berman and Plemmons, 1994), -1 is not an eigenvalue of $\boldsymbol{\Omega}$. Therefore $\mathbf{I} + \boldsymbol{\Omega}$ is invertible and $\mathbf{I} + \boldsymbol{\Omega}^\mathsf{T}$ is also invertible. Hence,

$$\hat{\mathbf{a}} = \left(\mathbf{I} + \boldsymbol{\Omega}^\mathsf{T}\right)^{-1}\mathbf{a}^*.$$

□

Proof of Proposition 3. (i) *Existence.* We shall build a sufficiently large box where the maximum is reached. Let $\mathbf{a} \geq \mathbf{0}$. Let us study the partial derivatives of SW. We have, for all l,

$$\begin{aligned}
\frac{\partial\ SW}{\partial\ a_l}(\mathbf{a}) &= p_l - q_l'\left(a_l + \sum_{k:k\neq l} \omega_{kl} a_k\right) - \sum_{j:j\neq l} \omega_{lj} q_j' \left(a_j + \sum_{i:i\neq j} \omega_{ij} a_i\right) \\
&\leq p_l - q_l'\left(a_l + \sum_{k:k\neq l} \omega_{kl} a_k\right), \text{ since } q_j \text{ is increasing} \\
&\leq p_l - q_l'(a_l), \text{ since } q_l \text{ is convex.}
\end{aligned}$$

So whenever, $a_l > a_l^*$, $\frac{\partial\ SW}{\partial\ a_l}(\mathbf{a}) < 0$. Define $\mathbf{a}^\# \geq \mathbf{0}$ by

$$a_l^\# = \begin{cases} a_l\,, & \text{if } a_l \leq a_l^*, \\ a_l^*\,, & \text{otherwise.} \end{cases}$$

By construction, $\mathbf{a}^\# \in \prod_{l=1}^n [0, a_l^*]$ and $SW(\mathbf{a}) \leq SW(\mathbf{a}^\#)$. So, being continuous SW reaches its maximum on $\prod_{l=1}^n [0, a_l^*]$, hence on $\mathbb{R}_+^n$.

Uniqueness. Let us prove that SW is strictly concave whenever $\mathbf{I} + \mathbf{\Omega}$ is invertible. We may ignore the linear part, we shall show that the following function c is strictly convex where

$$c(\mathbf{a}) = \sum_l q_l \left(a_l + \sum_{k:k\neq l} \omega_{kl} a_k\right), \ \mathbf{a} \geq \mathbf{0}\ .$$

Let $\mathbf{a}', \mathbf{a}'' \geq \mathbf{0}$ and $\theta \in (0,1)$ with $\mathbf{a}' \neq \mathbf{a}''$. Since $\mathbf{I} + \mathbf{\Omega}$ is invertible, $\mathbf{I} + \mathbf{\Omega}^\mathsf{T}$ is invertible too. Now, there exists some l_0 such that

$$\left(\left(\mathbf{I} + \mathbf{\Omega}^\mathsf{T}\right)\mathbf{a}'\right)_{l_0} \neq \left(\left(\mathbf{I} + \mathbf{\Omega}^\mathsf{T}\right)\mathbf{a}''\right)_{l_0}$$

that is

$$a_{l_0}' + \sum_{k:k\neq l_0} \omega_{kl_0} a_k' \neq a_{l_0}'' + \sum_{k:k\neq l_0} \omega_{kl_0} a_k''\ .$$

By strict convexity of q_{l_0} and convexity of q_l for $l \neq l_0$ it comes

$$\begin{aligned}
& q_{l_0}\left(\theta\left(a_{l_0}' + \sum_{k:k\neq l_0} \omega_{kl_0} a_k'\right) + (1-\theta)\left(a_{l_0}'' + \sum_{k:k\neq l_0} \omega_{kl_0} a_k''\right)\right) \\
< \ & \theta q_{l_0}\left(a_{l_0}' + \sum_{k:k\neq l_0} \omega_{kl_0} a_k'\right) + (1-\theta)\, q_{l_0}\left(a_{l_0}'' + \sum_{k:k\neq l_0} \omega_{kl_0} a_k''\right)
\end{aligned}$$

and

$$q_l\left(\theta\left(a_l' + \sum_{k:k\neq l} \omega_{kl} a_k'\right) + (1-\theta)\left(a_l'' + \sum_{k:k\neq l} \omega_{kl} a_k''\right)\right)$$

$$\leq \theta q_l\left(a_l' + \sum_{k:k\neq l} \omega_{kl} a_k'\right) + (1-\theta)\, q_l\left(a_l'' + \sum_{k:k\neq l} \omega_{kl} a_k''\right) .$$

Summing these inequalities over l, we obtain

$$c\left(\theta \mathbf{a}' + (1-\theta)\,\mathbf{a}''\right) < \theta c\left(\mathbf{a}'\right) + (1-\theta)\, c\left(\mathbf{a}''\right)$$

and this establishes strong convexity of c, thus strong concavity of SW. Therefore SW's maximum is unique.

Interiority. Let us show that A2 implies that $\mathbf{I}+\mathbf{\Omega}$ is invertible. Since $p_l = q_l'(a_l^*)$ for all l, A2 may be written

$$\left(\mathbf{I}-\mathbf{\Omega}\right)\mathbf{q}'\left(\mathbf{a}^*\right) - \mathbf{q}'\left(\mathbf{\Omega}^{\mathsf{T}}\mathbf{a}^*\right) >> \mathbf{0}$$

thus

$$\mathbf{q}'\left(\mathbf{a}^*\right) - \mathbf{q}'\left(\mathbf{\Omega}^{\mathsf{T}}\mathbf{a}^*\right) >> \mathbf{0}$$

and since q_l' is invertible and increasing for all l, we have

$$\mathbf{a}^* - \mathbf{\Omega}^{\mathsf{T}}\mathbf{a}^* >> \mathbf{0}$$

that is, A1 holds. Hence, $\rho(\mathbf{\Omega}) < 1$ and therefore, $\mathbf{I}+\mathbf{\Omega}$ is invertible.

Now, we show that the efficient profile is interior whenever A2 is satisfied. We know that $\tilde{\mathbf{a}} \in \prod_{l=1}^n [0, a_l^*]$. Under A2, we may prove a sharper statement.

Lemma. *Under A2, the efficient profile satisfies* $\left(\mathbf{I}+\mathbf{\Omega}^{\mathsf{T}}\right)\tilde{\mathbf{a}} \leq \mathbf{a}^*$.

Proof. Let $l \in \{1,\ldots,n\}$. If $\tilde{a}_l > 0$, the first order conditions give

$$q_l'\left(\tilde{a}_l + \sum_{k:k\neq l} \omega_{kl}\tilde{a}_k\right) = p_l - \sum_{j:j\neq l} \omega_{lj} q_j'\left(\tilde{a}_j + \sum_{i:i\neq j} \omega_{ij}\tilde{a}_i\right) \leq p_l .$$

Since q_l' is increasing, $\tilde{a}_l + \sum_{k:k\neq l}\omega_{kl}\tilde{a}_k \leq a_l^*$, that is $((\mathbf{I}+\mathbf{\Omega}^{\mathsf{T}})\tilde{\mathbf{a}})_l \leq a_l^*$. If $\tilde{a}_l = 0$, we have $\left(\left(\mathbf{I}+\mathbf{\Omega}^{\mathsf{T}}\right)\tilde{\mathbf{a}}\right)_l = \left(\mathbf{\Omega}^{\mathsf{T}}\tilde{\mathbf{a}}\right)_l \leq \left(\mathbf{\Omega}^{\mathsf{T}}\mathbf{a}^*\right)_l \leq a_l^*$ because A2 entails A1. □

Let us prove now that necessarily $\tilde{\mathbf{a}} >> \mathbf{0}$. Assume on the contrary that for some l, we have $\tilde{a}_l = 0$. We have,

$$\begin{aligned}
\frac{\partial\ SW}{\partial\ a_l}(\tilde{\mathbf{a}}) &= p_l - q_l'\left(0 + \sum_{k:k\neq l}\omega_{kl}\tilde{a}_k\right) - \sum_{j:j\neq l}\omega_{lj}q_j'\left(\tilde{a}_j + \sum_{i:i\neq j}\omega_{ij}\tilde{a}_i\right) \\
&\geq p_l - q_l'\left(\sum_{k:k\neq l}\omega_{kl}\tilde{a}_k\right) - \sum_{j:j\neq l}\omega_{lj}q_j'\left(a_j^*\right),
\end{aligned}$$

by the Lemma and since q_j' is increasing for all j

$$\begin{aligned}
&= p_l - q_l'\left(\sum_{k:k\neq l}\omega_{kl}\tilde{a}_k\right) - \sum_{j:j\neq l}\omega_{lj}p_j \\
&\geq p_l - q_l'\left(\sum_{k:k\neq l}\omega_{kl}a_k^*\right) - \sum_{j:j\neq l}\omega_{lj}p_j,
\end{aligned}$$

since $\tilde{a}_j \leq a_j^*$ for all j, and q_l' is increasing

$$= \left(\left(\mathbf{I} - \mathbf{\Omega}\right)\mathbf{p}\right)_l - q_l'\left(\left(\mathbf{\Omega}^{\mathsf{T}}\mathbf{a}^*\right)_l\right) > 0, \text{ by A2.}$$

This is in contradiction with the l^{th} first order condition of the efficient profile, i.e., $\frac{\partial\ SW}{\partial\ a_l}(\tilde{\mathbf{a}}) \leq 0$.

(ii) See (i). □

Proof of Proposition 4. Since $\tilde{a}_l > 0$ for all l whenever A2 is met, at social optimum we have the following first order conditions:

$$\forall l \in N,\ \ p_l - q_l'\left(\tilde{a}_l + \sum_{k:k\neq l}\omega_{kl}\tilde{a}_k\right) - \sum_{j:j\neq l}\omega_{lj}q_j'\left(\tilde{a}_j + \sum_{i:i\neq j}\omega_{ij}\tilde{a}_i\right) = 0.$$

Let $e_j = q_j'(\tilde{a}_j + \sum_{i:i\neq j}\omega_{ij}\tilde{a}_i)$ for all j. Then, the first order conditions may be written:

$$\forall l \in N,\ \ p_l = e_l + \sum_{j:j\neq l}\omega_{lj}e_j = \left(\left(\mathbf{I} + \mathbf{\Omega}\right)\mathbf{e}\right)_l = \left(\mathbf{I} + \mathbf{\Omega}\right)_{l.}\mathbf{e}.$$

In matrix notation,

$$\mathbf{p} = \left(\mathbf{I} + \mathbf{\Omega}\right)\mathbf{e}.$$

Under A2, $\mathbf{I}+\mathbf{\Omega}$ is invertible, so we obtain

$$\mathbf{e} = (\mathbf{I}+\mathbf{\Omega})^{-1}\,\mathbf{p}.$$

We have specified $e_j = q_j'(\tilde{a}_j + \sum_{i:i\neq j}\omega_{ij}\tilde{a}_i)$ for all j. Thus,

$$\forall j \in N,\;\; \left(q_j'\right)^{-1}(e_j) = \tilde{a}_j + \sum_{i:i\neq j}\omega_{ij}\tilde{a}_i = \left(\mathbf{I}+\mathbf{\Omega}^{\mathsf{T}}\right)_{j.}\tilde{\mathbf{a}}.$$

Hence,

$$(\mathbf{q}')^{-1}(\mathbf{e}) = \left(\mathbf{I}+\mathbf{\Omega}^{\mathsf{T}}\right)\tilde{\mathbf{a}},$$

and therefore,

$$\tilde{\mathbf{a}} = \left(\mathbf{I}+\mathbf{\Omega}^{\mathsf{T}}\right)^{-1}(\mathbf{q}')^{-1}(\mathbf{e}) = \left(\mathbf{I}+\mathbf{\Omega}^{\mathsf{T}}\right)^{-1}(\mathbf{q}')^{-1}\left((\mathbf{I}+\mathbf{\Omega})^{-1}\,\mathbf{p}\right).$$

□

Proof of Proposition 5. Under A0 and A2, $\tilde{a}_l > 0$ for all l, thus at social optimum we have the following first order conditions:

$$\forall l \in N,\;\; p_l - q_l'\left(\tilde{a}_l + \sum_{k:k\neq l}\omega_{kl}\tilde{a}_k\right) - \sum_{j:j\neq l}\omega_{lj}q_j'\left(\tilde{a}_j + \sum_{i:i\neq j}\omega_{ij}\tilde{a}_i\right) = 0.$$

Then, by strict convexity of the cost functions, $p_l - q_l'(\tilde{a}_l + \sum_{k:k\neq l}\omega_{kl}\tilde{a}_k) \geq 0$.

Now, we show that the efficient profile $\tilde{\mathbf{a}}$ is also the Nash equilibrium of a game where each player is constrained to exert an action at most equal to her efficient level. Given $\mathbf{\Omega}$ and $\tilde{\mathbf{a}}_{-l}$, a player l's maximization program is:

$$\max_{a_l}\; p_l a_l - q_l\left(a_l + \sum_{k:k\neq l}\omega_{kl}\tilde{a}_k\right)$$
$$\text{s.t.}\quad a_l \in [0, \tilde{a}_l].$$

By assumption, $U_l(\tilde{\mathbf{a}}_{-l}, a_l) = p_l a_l - q_l(a_l + \sum_{k:k\neq l}\omega_{kl}\tilde{a}_k)$ is a strictly concave payoff function and

$$U_l'(\tilde{\mathbf{a}}_{-l}, a_l) = p_l - q_l'\left(a_l + \sum_{k:k\neq l}\omega_{kl}\tilde{a}_k\right).$$

Then, for $a_l = \tilde{a}_l$, we have

$$U_l'(\tilde{\mathbf{a}}) = p_l - q_l'\left(\tilde{a}_l + \sum_{k:k\neq l}\omega_{kl}\tilde{a}_k\right) \geq 0$$

because $\tilde{\mathbf{a}}$ is the efficient profile of game $\mathcal{G}(\mathbf{\Omega}, \mathbf{p}, \mathbf{q})$. Since U_l is strictly concave and $U_l'(\tilde{\mathbf{a}}) \geq 0$, $a_l = \tilde{a}_l$ is player l's best reply. By Proposition 4, we obtain the result. □

Proof of Proposition 6. (i) Under A0 and A2, $\tilde{a}_l > 0$ for all l. Thus, at social optimum we have the following first order conditions:

$$\forall l \in N, \;\; p_l - q_l' \left(\tilde{a}_l + \sum_{k:k \neq l} \omega_{kl} \tilde{a}_k \right) - \sum_{j:j \neq l} \omega_{lj} q_j' \left(\tilde{a}_j + \sum_{i:i \neq j} \omega_{ij} \tilde{a}_i \right) = 0.$$

Let $\tau_l = \frac{1}{p_l} \sum_{j:j \neq l} \omega_{lj} q_j' (\tilde{a}_j + \sum_{i:i \neq j} \omega_{ij} \tilde{a}_i)$ for all l. So $\tau_l \geq 0$ for all l. The first order conditions may be written:

$$\forall l \in N, \;\; p_l \left(1 - \tau_l\right) = q_l' \left(\tilde{a}_l + \sum_{k:k \neq l} \omega_{kl} \tilde{a}_k \right) > 0,$$

thus $\tau_l < 1$. Then, the efficient profile $\tilde{\mathbf{a}}$ is also a Nash equilibrium of a game where, for all l,

$$U_l \left(\mathbf{a}\right) = p_l \left(1 - \tau_l\right) a_l - q_l \left(a_l + \sum_{k:k \neq l} \omega_{kl} a_k \right).$$

Let $e_j = q_j' (\tilde{a}_j + \sum_{i:i \neq j} \omega_{ij} \tilde{a}_i)$ for all j. Then,

$$\forall l \in N, \;\; \tau_l = \frac{1}{p_l} \sum_{j:j \neq l} \omega_{lj} e_j = \frac{1}{p_l} \; \boldsymbol{\Omega}_{l.} \mathbf{e}.$$

Since $\mathbf{e} = \left(\mathbf{I} + \boldsymbol{\Omega}\right)^{-1} \mathbf{p}$ (see the proof of Proposition 4), it follows that

$$\forall l \in N, \;\; \tau_l = \frac{1}{p_l} \; \boldsymbol{\Omega}_{l.} \left(\mathbf{I} + \boldsymbol{\Omega}\right)^{-1} \mathbf{p} = \frac{1}{p_l} \left(\boldsymbol{\Omega} \left(\mathbf{I} + \boldsymbol{\Omega}\right)^{-1} \mathbf{p} \right)_l.$$

Finally, we note that

$$\boldsymbol{\Omega} \left(\mathbf{I} + \boldsymbol{\Omega}\right)^{-1} = \mathbf{I} - \left(\mathbf{I} + \boldsymbol{\Omega}\right)^{-1}$$

so

$$\forall l \in N, \;\; \tau_l = \frac{p_l - \left(\mathbf{b}^{+}_{\text{alt}} \left(\boldsymbol{\Omega}, \mathbf{p}\right)\right)_l}{p_l}$$

(ii) We have

$$\tilde{\boldsymbol{\tau}} \geq \mathbf{0} \iff \mathbf{p} - \mathbf{b}^{+}_{\text{alt}} \left(\boldsymbol{\Omega}, \mathbf{p}\right) \geq \mathbf{0} \iff \mathbf{p} - \left(\mathbf{I} + \boldsymbol{\Omega}\right)^{-1} \mathbf{p} \geq \mathbf{0}.$$

Now, observe that

$$\mathbf{p} - \left(\mathbf{I} + \boldsymbol{\Omega}\right)^{-1} \mathbf{p} = \left(\mathbf{I} - \left(\mathbf{I} + \boldsymbol{\Omega}\right)^{-1} \right) \mathbf{p} = \boldsymbol{\Omega} \left(\mathbf{I} + \boldsymbol{\Omega}\right)^{-1} \mathbf{p} = \boldsymbol{\Omega} \mathbf{b}^{+}_{\text{alt}} \left(\boldsymbol{\Omega}, \mathbf{p}\right),$$

so

$$\forall l \in N, \;\; \tilde{\tau}_l > 0 \iff \left(\mathbf{\Omega} \mathbf{b}^{+}_{\text{alt}}\left(\mathbf{\Omega}, \mathbf{p}\right)\right)_l > 0.$$

Let us show that $\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega}, \mathbf{p}) >> \mathbf{0}$ whenever A2 is met. We have

$$\forall l \in N, \;\; \tilde{\tau}_l = \frac{p_l - \left(\mathbf{b}^{+}_{\text{alt}}\left(\mathbf{\Omega}, \mathbf{p}\right)\right)_l}{p_l} < 1 \iff \mathbf{b}^{+}_{\text{alt}}\left(\mathbf{\Omega}, \mathbf{p}\right) >> \mathbf{0}.$$

Now, observe that

$$\mathbf{b}^{+}_{\text{alt}}\left(\mathbf{\Omega}, \mathbf{p}\right) = \mathbf{b}^{-}_{\text{alt}}\left(\mathbf{\Omega}^{\mathsf{T}}, \mathbf{p}\right).$$

By Proposition 2, we have

$$\left(\mathbf{I} - \mathbf{\Omega}\right)\mathbf{p} >> \mathbf{0} \Rightarrow \mathbf{b}^{-}_{\text{alt}}\left(\mathbf{\Omega}^{\mathsf{T}}, \mathbf{p}\right) >> \mathbf{0},$$

and since $q'_l > 0$ for all l, we obtain

$$\left(\mathbf{I} - \mathbf{\Omega}\right)\mathbf{p} - \mathbf{q}'\left(\mathbf{\Omega}^{\mathsf{T}}\mathbf{a}^*\right) >> \mathbf{0} \Rightarrow \left(\mathbf{I} - \mathbf{\Omega}\right)\mathbf{p} >> \mathbf{0},$$

that is,

$$\text{A2} \Rightarrow \mathbf{b}^{+}_{\text{alt}}\left(\mathbf{\Omega}, \mathbf{p}\right) >> \mathbf{0}.$$

Hence,

$$\forall l \in N, \;\; \tilde{\tau}_l > 0 \iff \left(\mathbf{\Omega} \mathbf{b}^{+}_{\text{alt}}\left(\mathbf{\Omega}, \mathbf{p}\right)\right)_l > 0 \iff \exists j \in N \;/\; \omega_{lj} > 0.$$

□

Proof of Proposition 7. Let us write $\forall l, \;\; b_l = (\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega}, \mathbf{p}))_l$ as a shorthand. From Proposition 6, we have for all l that

$$0 \leq \tilde{\tau}_{\min} \leq \frac{p_l - b_l}{p_l} \leq \tilde{\tau}_{\max} < 1$$

or equivalently

$$\left(1 - \tilde{\tau}_{\min}\right)^{-1} b_l \leq p_l \leq \left(1 - \tilde{\tau}_{\max}\right)^{-1} b_l.$$

Since $\mathbf{I} + \mathbf{\Omega}$ is nonnegative,

$$\left(1 - \tilde{\tau}_{\min}\right)^{-1} \mathbf{p} \leq \left(\mathbf{I} + \mathbf{\Omega}\right)\mathbf{p} \leq \left(1 - \tilde{\tau}_{\max}\right)^{-1} \mathbf{p}$$

or equivalently

$$\left(1 + \frac{\tilde{\tau}_{\min}}{1 - \tilde{\tau}_{\min}}\right)\mathbf{p} \leq \left(\mathbf{I} + \mathbf{\Omega}\right)\mathbf{p} \leq \left(1 + \frac{\tilde{\tau}_{\max}}{1 - \tilde{\tau}_{\max}}\right)\mathbf{p}$$

so

$$\frac{\tilde{\tau}_{\min}}{1-\tilde{\tau}_{\min}}\mathbf{p} \le \mathbf{\Omega p} \le \frac{\tilde{\tau}_{\max}}{1-\tilde{\tau}_{\max}}\mathbf{p}.$$

Then, by Theorem 2.1.11 p.28 in Berman and Plemmons (1994), since $\mathbf{p} >> \mathbf{0}$ we have

$$\frac{\tilde{\tau}_{\min}}{1-\tilde{\tau}_{\min}} \le \rho\left(\mathbf{\Omega}\right) \le \frac{\tilde{\tau}_{\max}}{1-\tilde{\tau}_{\max}},$$

that is,

$$\tilde{\tau}_{\min} \le \frac{\rho\left(\mathbf{\Omega}\right)}{1+\rho\left(\mathbf{\Omega}\right)} \le \tilde{\tau}_{\max}.$$

□

Proof of Proposition 8. Let us write $\forall l,\ b_l = (\mathbf{b}^{+}_{\text{alt}}(\mathbf{\Omega},\mathbf{p}))_l$ as a shorthand.

(i) Assume the optimal tax rate is uniform, i.e., $\forall l,\ \tilde{\tau}_l = \tilde{\tau}$. Then,

$$\begin{array}{rrcl} & \tilde{\tau} & = & \frac{p_l-b_l}{p_l},\ \forall l \\ \iff & \left(1-\tilde{\tau}\right)^{-1}\mathbf{b} & = & \mathbf{p} \\ \iff & \left(1+\frac{\tilde{\tau}}{1-\tilde{\tau}}\right)\mathbf{p} & = & \left(\mathbf{I}+\mathbf{\Omega}\right)\mathbf{p} \\ \iff & \frac{\tilde{\tau}}{1-\tilde{\tau}}\mathbf{p} & = & \mathbf{\Omega p} \end{array}$$

(ii) If $\mathbf{\Omega} = \mathbf{0}$, then 0, the unique eigenvalue, is clearly an optimal uniform tax rate. If $\mathbf{\Omega} \neq \mathbf{0}$, assume $\mathbf{p}$ is an eigenvector of $\mathbf{\Omega}$ and let $t_{\mathbf{p}} \in \mathbb{C}$ be its associated eigenvalue. Since $t_{\mathbf{p}}\mathbf{p} = \mathbf{\Omega p}$, $\mathbf{\Omega} \ge \mathbf{0}$ and $\mathbf{p} >> 0$, for some l, $t_{\mathbf{p}}p_l > 0$, thus $t_{\mathbf{p}} > 0$. We have

$$\left(\mathbf{I}+\mathbf{\Omega}\right)\mathbf{p} = \left(1+t_{\mathbf{p}}\right)\mathbf{p}.$$

Thus,

$$\mathbf{p} = \left(\mathbf{I}+\mathbf{\Omega}\right)^{-1}\left(1+t_{\mathbf{p}}\right)\mathbf{p} = \left(1+t_{\mathbf{p}}\right)\mathbf{b},$$

so for all l,

$$\tilde{\tau}_l = 1-\frac{b_l}{p_l} = 1-\frac{\frac{1}{1+t_{\mathbf{p}}}p_l}{p_l} = 1-\frac{1}{1+t_{\mathbf{p}}} = \frac{t_{\mathbf{p}}}{1+t_{\mathbf{p}}}.$$

(iii) If $\mathbf{p}$ is an eigenvector of $\mathbf{\Omega}$, then by Corollary 2.1.12 p.28 in Berman and Plemmons (1994), $\mathbf{p}$ corresponds to $\rho(\mathbf{\Omega})$ since $\mathbf{p} >> \mathbf{0}$. So,

$$t_{\mathbf{p}} = \rho\left(\mathbf{\Omega}\right) \iff \tilde{\tau} = \frac{\rho\left(\mathbf{\Omega}\right)}{1+\rho\left(\mathbf{\Omega}\right)}.$$

□

B. Corner solutions

Consider a unique equilibrium where some players are inactive. This may be the case when A1 does not hold but $\mathbf{I}+\boldsymbol{\Omega}$ is a P-matrix. Let $C=\{i:\hat{a}_i>0\}$ and $\overline{C}$ its complement. We note $\boldsymbol{\Omega}_{C\times C}$ the adjacency matrix of the subnetwork obtained by deleting all the inactive players at equilibrium. The subvector $\hat{\mathbf{a}}_C$ consisting of all the active players in the original game is also a Nash equilibrium for the subgame obtained. Moreover, there are no inactive players in this subgame, hence $\hat{\mathbf{a}}_C$ is equal to the individual power measure of the subnetwork obtained after deleting the inactive players, provided that $(\mathbf{I}+\boldsymbol{\Omega})_{C\times C}$ is invertible (guaranteed if $\mathbf{I}+\boldsymbol{\Omega}$ is a P-matrix).

Next, consider a unique social optimum where some players are inactive. This may be the case when A2 does not hold but $\mathbf{I}+\boldsymbol{\Omega}$ is invertible. Let $D=\{i:\tilde{a}_i>0\}$ and $\overline{D}$ its complement. The first order conditions of social welfare maximization for the active players are

$$\begin{aligned}\forall l\in D,\ \ p_l-q_l'\left(\tilde{a}_l+\sum_{k:k\neq l,k\in D}\omega_{kl}\tilde{a}_k\right)-\sum_{j:j\neq l,j\in D}\omega_{lj}q_j'\left(\tilde{a}_j+\sum_{i:i\neq j,i\in D}\omega_{ij}\tilde{a}_i\right)\\ -\sum_{g:g\neq l,g\in\overline{D}}\omega_{lg}q_g'\left(0+\sum_{h:h\neq g,h\in D}\omega_{hg}\tilde{a}_h\right)=0.\end{aligned}$$

We note $\boldsymbol{\Omega}_{D\times D}$ the submatrix obtained by deleting all the inactive players at efficiency and $\boldsymbol{\Omega}_{D\times\overline{D}}$ the (possibly rectangular) submatrix of $\boldsymbol{\Omega}$ consisting of rows with all the active players and columns with all the inactive players. Assuming that $(\mathbf{I}+\boldsymbol{\Omega})_{D\times D}$ is invertible (guaranteed if $\mathbf{I}+\boldsymbol{\Omega}$ is a P-matrix), the subvector $\tilde{\mathbf{a}}_D$ consisting of all the active players at efficiency is "almost equal" to the social power measure of the subnetwork obtained after deleting the inactive players.

Property 2. *Let $\mathcal{G}(\boldsymbol{\Omega},\mathbf{p},\mathbf{q})$ be a network game. Under A0 and if $\mathbf{I}+\boldsymbol{\Omega}$ is a P-matrix, the equilibrium profile exists, is unique and is given by*

$$\begin{cases}\hat{\mathbf{a}}_{\overline{C}} &= \mathbf{0}\\ \hat{\mathbf{a}}_C &= \mathbf{b}^-_{\text{alt}}\left(\boldsymbol{\Omega}_{C\times C},(\mathbf{q}')^{-1}(\mathbf{p}_C)\right).\end{cases}$$

Moreover, the efficient profile exists, is unique and is a fixed point, i.e.,

$$\begin{cases}\tilde{\mathbf{a}}_{\overline{D}} &= \mathbf{0}\\ \tilde{\mathbf{a}}_D &= \mathbf{b}^-_{\text{alt}}\left(\boldsymbol{\Omega}_{D\times D},(\mathbf{q}')^{-1}\left(\mathbf{b}^+_{\text{alt}}\left(\boldsymbol{\Omega}_{D\times D},\boldsymbol{\phi}\left(\tilde{\mathbf{a}}_D\right)\right)\right)\right)\end{cases}$$

where $\boldsymbol{\phi}(\tilde{\mathbf{a}}_D)=\mathbf{p}_D-\boldsymbol{\Omega}_{D\times\overline{D}}\ \mathbf{q}'(\boldsymbol{\Omega}_{D\times\overline{D}}{}^{\mathsf{T}}\ \tilde{\mathbf{a}}_D)$.

Note that $\tilde{\mathbf{a}}_D$ may be obtained by solving the fixed point equation. Therefore, our policy results might be extended to corner (or partially-corner) equilibria and social optima. This is, however, a non-trivial issue.

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